Lecture 1 : Complex Number

$\sqrt{-1} = i \qquad \alpha^2 \geq 0$

$\sqrt{-2} = \sqrt{2}i$ ← imaginary unit

$\sqrt{2}$: imaginary part ($\mathrm{Im}\, z$)

$1+2i$: real part + imaginary part

$\mathbb{R} : a+0i, \; x$

$\mathbb{C} : a+bi, \; x+yi$ ($\mathbb{C}$: complex number)

Note
- complex number is a vector
- We can use every vector technique to deal with complex number.

ex) $\mathrm{Re}\, z = 1$, $\mathrm{Im}\, z = 2$

$0+bi$ → purely imaginary number.

$z_1 = 5-2i \; ; \; z_2 = 1+\frac{\sqrt{3}}{4}i$

ex1 $z_1+z_2 = (5-2i)+\left(1+\frac{\sqrt{3}}{4}i\right)$

$= (5+1)+\left(-2i+\frac{\sqrt{3}}{4}i\right) = 6 + \left(\frac{\sqrt{3}}{4}-2\right)i$

6 : real part; $\left(\frac{\sqrt{3}}{4}-2\right)$: imaginary part; i : imaginary unit

ex2 $-2z_1 = -2(5-2i) = -10+4i$

ex3 $z_1 z_2 = (5-2i)\left(1+\frac{\sqrt{3}}{4}i\right) = 5\left(1+\frac{\sqrt{3}}{4}i\right) - 2i\left(1+\frac{\sqrt{3}}{4}i\right)$

$= 5+\frac{5\sqrt{3}}{4}i - 2i - \frac{\sqrt{3}}{2}i$

$= \frac{10+\sqrt{3}}{2} + \left(\frac{5\sqrt{3}}{4}-2\right)i$

ex4 $\frac{z_1}{z_2} = \frac{5-2i}{1+\frac{\sqrt{3}}{4}i}$

$= \frac{5-2i}{1+\frac{\sqrt{3}}{4}i} \cdot \frac{1-\frac{\sqrt{3}}{4}i}{1-\frac{\sqrt{3}}{4}i} = \frac{5\left(1-\frac{\sqrt{3}}{4}i\right) - 2i\left(1-\frac{\sqrt{3}}{4}i\right)}{1^2-\left(\frac{\sqrt{3}}{4}\right)^2}$

$= \frac{5-\frac{5\sqrt{3}}{4}i - 2i - \frac{\sqrt{3}}{2}}{1-\frac{3}{16}}$

$= \frac{16\left(5-\frac{\sqrt{3}}{2}\right) - 16\left(\frac{5\sqrt{3}}{4}+2\right)i}{13}$

$= \frac{60-8\sqrt{3}}{13} - \frac{20\sqrt{3}+32}{13}i$

$z \in \mathbb{R} \qquad z = 1, 2 \cdots$

$z \in \mathbb{C} \qquad z = a+ib = x+yi = 1+2i$

* $\bar{z} = a-ib$ → conjugate

$z\bar{z} = (a+ib)(a-ib) = a^2+b^2$

* $|z| = \sqrt{z\bar{z}} = \sqrt{a^2+b^2}$ → modulus

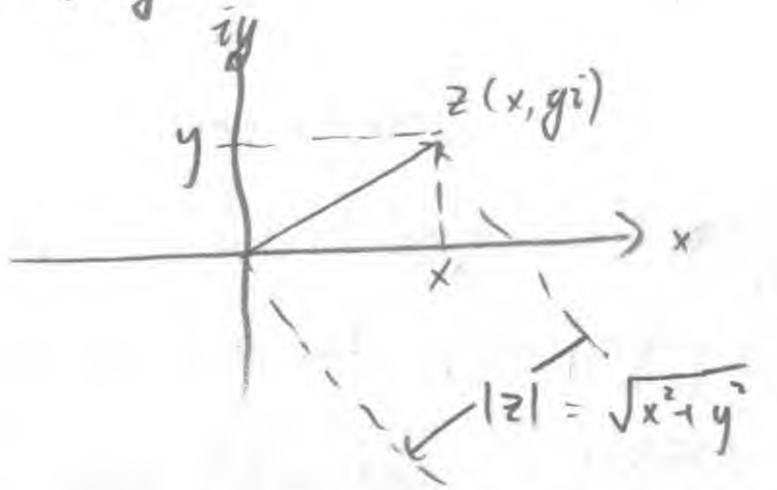
$z = 0 \Rightarrow$ real part and imaginary part all equal to 0.

$= 0+0i$

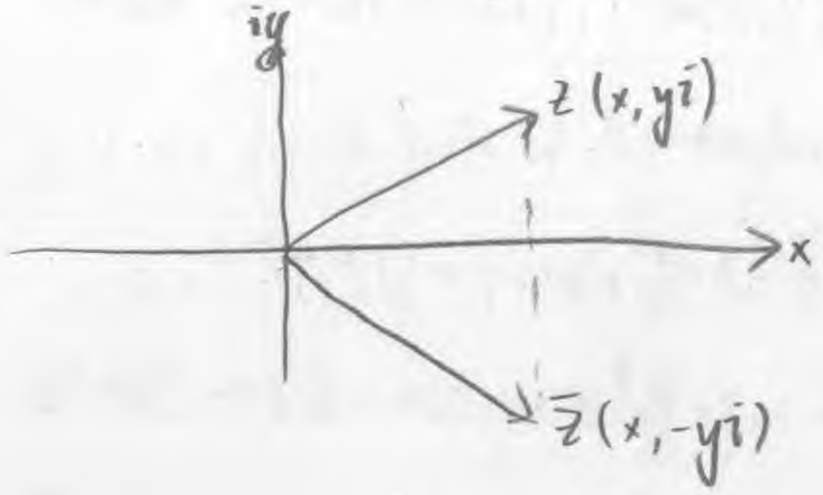
$|z| = 0 \iff z = 0+i0$.

$|z| > 0$ if $z \neq 0+i0$.

$z = x + yi$

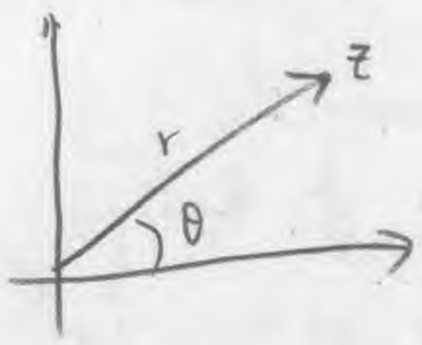


modulus = the length of $z = |z|$



$(x, y) \longrightarrow (r, \theta)$

$r = |z| = \sqrt{x^2+y^2}$; $\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right) = \arg(z)$



polar form

$z = r\cos\theta + ir\sin\theta$

$= r(\cos\theta + i\sin\theta) = r\,\mathrm{cis}\,\theta$

$z: \longrightarrow r = |z|,\ \theta = \arg z$

ex. $z = \sqrt{3} + i$

$r = |z| = \sqrt{z\bar{z}} = \sqrt{(\sqrt{3}+i)(\sqrt{3}-i)} = \sqrt{3+1} = 2$

$\tan\theta = \frac{1}{\sqrt{3}}$

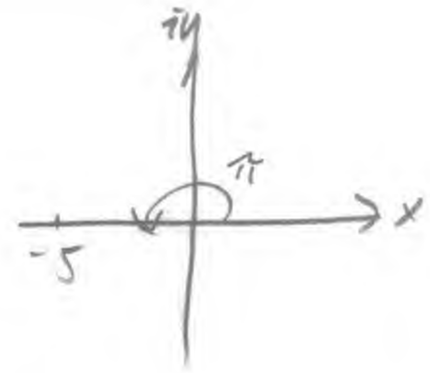
$\theta = \frac{\pi}{6} + 2k\pi,\ k = 0, 1, 2, \ldots$

$z = 2\left[\cos\left(\frac{\pi}{6} + 2k\pi\right) + i\sin\left(\frac{\pi}{6} + 2k\pi\right)\right]$.

$z = -5 \longrightarrow z = -5 + i0$

(r) $|z| = \sqrt{(-5)^2 + (0)^2} = \sqrt{25} = 5$

(r must be "positive" because it is the length of z.)



$\theta = \pi + 2k\pi$

$z = 5\left[\cos(\pi + 2k\pi) + i\sin(\pi + 2k\pi)\right]$

* $z_1 z_2 = r_1 r_2\left[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\right]$

where $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

* $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]$

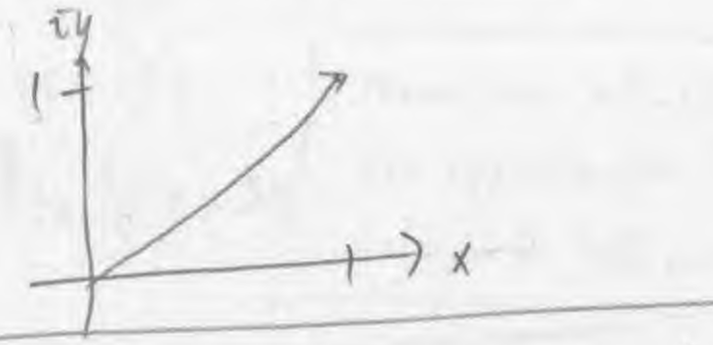
** $z^n = r^n(\cos n\theta + i\sin n\theta)$ Important!

DeMoivre's Theorem

$z = 1+i$, find z^{10}

$r = |z| = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta = \frac{\pi}{4}$



* If the question ask you show the polar form, $2k\pi$ is needed!

$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$z^{10} = (\sqrt{2})^{10}\left[\cos 10\left(\frac{\pi}{4}\right) + i\sin 10\left(\frac{\pi}{4}\right)\right]$

$= 32\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right) = 32\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right)$

$= 32\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$= 32i$.

$\sin 2\theta = 2\sin\theta\cos\theta$

$\sin 3\theta = ?$

$\sin 5\theta = ?$

$\sinh\theta = ?$

ex1 $(\cos\theta + i\sin\theta)^2 = \cos^2\theta + 2i\sin\theta\cos\theta + (i^2)\sin^2\theta$ ($i^2 = -1$)

$= (\cos^2\theta - \sin^2\theta) + (2\sin\theta\cos\theta)i$

$= \cos 2\theta + \sin 2\theta\, i$

$\therefore \cos 2\theta = \cos^2\theta - \sin^2\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

ex2 Find $\cos 5\theta$

By De Moivre's Theorem and Binomial Theorem, we have

$(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$

$(\cos\theta + i\sin\theta)^5$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta + 10i^2\cos^3\theta\sin^2\theta + 10i^3\cos^2\theta\sin^3\theta + 5i^4\cos\theta\sin^4\theta + i^5\sin^5\theta$

$= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

$= (\cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta) + i(5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta)$

$\therefore \cos 5\theta = \cos^5\theta - 10\cos^3\theta + 5\cos\theta\sin^4\theta$

Exponential form of complex number

$e^x \longrightarrow e^z = f(z) \in \mathbb{C}$

$\uparrow \quad z \in \mathbb{C}$

$z = x + yi$

$e^z = e^{x+yi} = e^x(\cos y + i\sin y)$ we know $y = \theta$

(e^x: r, θ)

ex1 $z = -2 + \frac{\pi}{3}i$

$e^z = e^{-2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

ex2 $z = 2 + 3i$

$e^z = e^2(\cos 3 + i\sin 3)$

ex3 $z = 0 + iy$

$e^z = e^0(\cos y + i\sin y) = \cos y + i\sin y$

$z = x + yi \longrightarrow r(\cos\theta + i\sin\theta) \longrightarrow re^{i\theta}$

Don't write the final answer like that!

$|e^z| = |e^x(\cos\theta + i\sin\theta)| = e^x|\cos\theta + i\sin\theta|$

$= e^x\sqrt{\cos^2\theta + \sin^2\theta} = e^x$

$|e^{i\theta}| = 1$

$e^{z_1+z_2} = e^{z_1}e^{z_2}$

$e^{-z} = \frac{1}{e^z}$

$(e^z)^n = e^{nz}$

$z_k = r^{\frac{1}{n}} e^{i\left(\frac{\theta+2k\pi}{n}\right)}$, where $k = 0, \dots, n-1$

★ $w = r(\cos\theta + i\sin\theta) = re^{i\theta}$ ← Don't write the final answer like that

$z_k = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta+2k\pi}{n}\right) + i\sin\left(\frac{\theta+2k\pi}{n}\right)\right]$

Find the sixth roots of the complex number $1+3i$.

$z = \sqrt[6]{1+3i} = \left[2(\cos\theta + i\sin\theta)\right]^{\frac{1}{6}}$

$= \sqrt[6]{2}\left(\cos\frac{2k\pi+\theta}{6} + i\sin\frac{2k\pi+\theta}{6}\right)$, where $\theta = \tan^{-1}(3)$

Find the fourth roots of the complex number $1+i$.

$z = \sqrt[4]{1+i} = \sqrt{2}^{\frac{1}{4}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{\frac{1}{4}}$

$= 2^{\frac{1}{8}}\left(\cos\frac{2k\pi+\pi}{4} + i\sin\frac{2k\pi+\pi}{4}\right)$

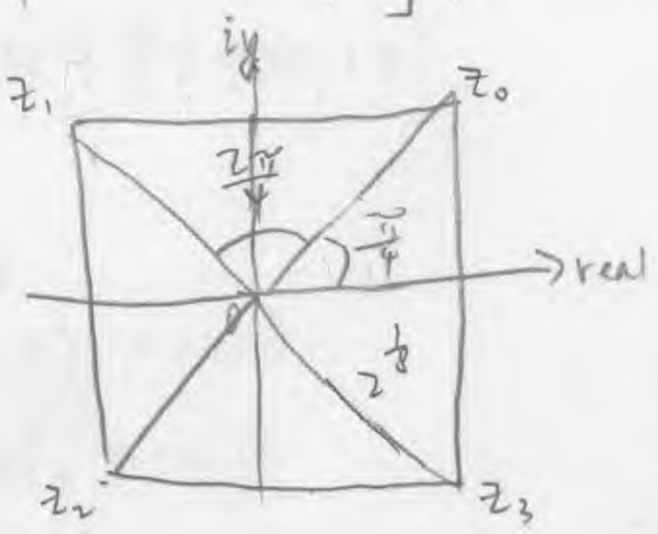
$= 2^{\frac{1}{8}}\left[\cos\frac{(2k+1)\pi}{4} + i\sin\frac{(2k+1)\pi}{4}\right]$, $k = 0, 1, 2, 3$

i.e. $z_0 = 2^{\frac{1}{8}}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$z_1 = 2^{\frac{1}{8}}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

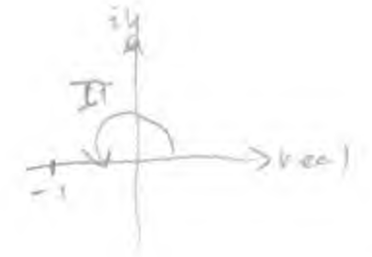
$z_2 = 2^{\frac{1}{8}}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

$z_3 = 2^{\frac{1}{8}}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$



Find all roots of the complex number.

$w = -32$; $z^5 = w$



$z^5 = -32$

$z = (-32)^{\frac{1}{5}} = (32)^{\frac{1}{5}}(-1)^{\frac{1}{5}} = 2\left(\cos\pi + i\sin\pi\right)^{\frac{1}{5}}$

$= 2\left(\cos\frac{2k\pi+\pi}{5} + i\sin\frac{2k\pi+\pi}{5}\right)$,

$k = 0, 1, 2, 3, 4$.

$z_0 = 2\left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)$

$z_1 = 2\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$

$z_2 = 2\left(\cos\frac{5\pi}{5} + i\sin\frac{5\pi}{5}\right) = 2(-1) = -2$

$z_3 = 2\left(\cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}\right)$

$z_4 = 2\left(\cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}\right)$

Never use calculator to get the approximation. The answers need to be 100% correct!

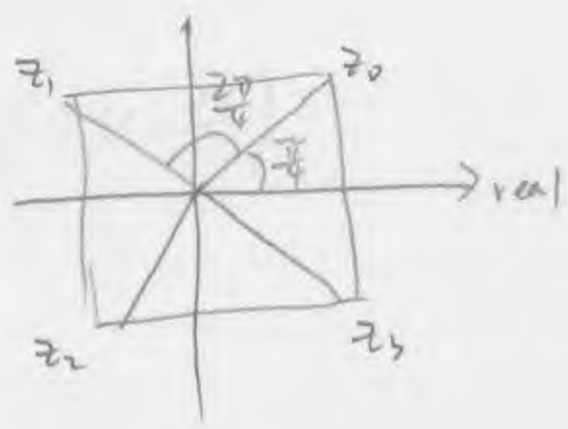
ex $z^4 = -1$

$z = (-1)^{\frac{1}{4}} = \left(\cos\pi + i\sin\pi\right)^{\frac{1}{4}} = \cos\frac{2k\pi+\pi}{4} + i\sin\frac{2k\pi+\pi}{4}$,

$k = 0, 1, 2, 3$.

$z_0 = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

$z_1 = \cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4} = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i$

$z_2 = \cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$

$z_3 = \cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4} = \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}i$